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The Present Status of the Coish Model

HEDLEY C. MORRIS

School of Mathematics, Trinity College Dublin, Dublin 2, Ireland

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Abstract

We present a review of the Coish model for a non-metric world the points of which are coordinatised by elements from a finite field. The model is brought up to date by introducing recently determined results on the irreducible representations of the special linear group over a finite field due to Tanaka.

1. *Introduction*

Of all discrete models two particular cases separate themselves out as special. One is the gaussian integer approach to the discrete integral lattice due to Schild (1949) and the other is the Coish model (Coish, 1959; Shapiro, 1960; Joos, 1964; Ahmavaara, 1965–66). They stand apart because they are truely non-metric models which have been developed to a point when we can consider the possibility of constructing physical theories on them. In the following sections we provide a review of the Coish model which follows roughly its historical development up to 1964. Then, in a final section, we try to bring the model up to date by observing that certain mathematical results, unknown in 1964, have since become available and allow us to complete the model. The missing mathematical results were the representations of $SL(2, GF(p))$, the special linear group over the Galois field of order p, and did not become available until 1967 [11–12]. The dimensions of the irreducible representations found by Tanaka for $SL(2, GF(p))$ are $\frac{1}{2}(p-1)$, $(p-1)$, p, $\frac{1}{2}(p+1)$, $(p+1)$ and so there are no low dimensional irreducible representations at all. This means that if we make the normal group dynamical interpretation of the dimension as connected with spin we arrive at the unfortunate conclusion that there can be no low spin particles. This would seem to rule the model out on physical grounds provided we maintain this interpretation. However, the model remains of great didactic value and of considerable aesthetic appeal.

2. *The Ordering*

Coish observes that many of the problems encountered in physics result from the use of an infinite ground field. He decides to investigate the possibility of an event space in which the four coordinates are not drawn from the real number field as usual but from a ring. This ring he terms the world ring. Of the many possibilities he chooses the simplest, a finite field. These are all classified and are simply $GF(p^n)$ where $GF(p^n)$ is the field of integers mod (p^n) and n an integer. What must be done is to find such a finite field, a sufficiently large portion of which is 'like' the real number system. We must fill in the gap between 10^{-13} and 10^{27}.

By 'like the real number system' we mean that such a subset must be transitively ordered, a very non-trivial constraint. The general problem of finite geometries which can be made to approximate the Euclidean plane so closely as may be desired was considered in detail by Jarnfelt (1951) and Kustaanheimo (1950). Clearly there is no difficulty in finding enough points from a field $GF(p)$ provided p is chosen large enough. In fact to fill in the range 10^{-13}–10^{27} a prime $p \sim 10^{10^{81}}$ should prove sufficient. The ordering is the more difficult problem being due to the existence of only trivial valuations on finite fields. It is possible, however, to use a very similar construct to that used for the real numbers. Any multiplicative group G can be ordered if one can find an onto homomorphism $\tau : G \to \{1, -1\}$. If one then defines C^+ as kert and calls this the positive cone, the complement C^- is clearly isomorphic and we call this the negative cone. One can then define,

$$a > b \equiv a - b \in C^+$$

$$b < a \equiv b - a \in C^-$$

This is the case with the real numbers where C^+, C^- are the normal cones of positive and negative reals defined by the construction of the real number system. All that is required is some fundamental property which an element does or does not have without exception.

In the case of a finite field $GF(q)$ it can be shown that there exist elements p known as primitive elements processing property that the smallest power of p equal to the unit element is $q - 1, p^{q-1} = 1$ (and there is no lesser power having this value). This being the case it is clear that p generates the multiplicative group and consequently each element is a power either even or odd, of p.

The even-odd property is just what is required and so we have the homomorphism

$$\tau(g) = 1 \qquad g = p^{2n}$$

$$= -1 \qquad g = p^{2n+1}$$

Thus we define

$$a_q > 0_q \quad \text{iff} \quad a_q \in C_\tau^+$$

$$a_q < 0_q \quad \text{iff} \quad a_q \in C_\tau^-$$

and

$$a_q > b_q \quad \text{iff} \quad (a-b)_q > 0_q$$

$$a_q < b_q \quad \text{iff} \quad (a-b)_q < 0_q$$

The transitivity condition is more difficult. Consider for example,

$$0_q < 1_q < 2q$$

This is transitively ordered if in addition to the trivial

$$2_q - 1_q = 1_q - 0_q = 1_q > 0_q$$

also

$$2_q - 0_q = 2_q > 0_q$$

which for $q = 5$ is not the case.

Kustaanheimo has shown that if the prime is chosen to have the form

$$p = \left(8x \prod_{i=1}^{k} q_i - 1\right)$$

where x is an odd integer and $\Pi_i\, q_i$ the produce of the first k-odd primes, then -1_q is 'negative' and 2 and the first k-odd primes are 'positive'. For such a prime the first N integers for large N can be transitively ordered and consequently the geometry would appear to be like ordinary Euclidean plane up to very large and down to very small distances.

3. *Lorentz Type Groups* (Joos, 1964; Ahmavaara, 1965–66)

If we are going to adopt the purely group theoretic approach to physics we must construct the relativity groups over this world ring $GF(p)$ where p is a K-prime. Let us introduce the following notation which is standard for the subgroups of the group of transformations $(\Lambda, a) : x \to \Lambda x + a$ where Λ is a Lorentz matrix with entries from $GF(p)$.

$\mathscr{T} = \{(x, a) \mid a \in GF(p)\}$	Translation group
$\mathscr{L} = \{(\Lambda, 0) \mid \tilde{\Lambda} g \Lambda = g\}$	Lorentz group
$\mathscr{C} = \{(\Lambda, 0) \mid \tilde{\Lambda} g \Lambda = \pm g\}$	Coish group
$\mathscr{P} = \{(\Lambda, a) \mid \tilde{\Lambda} g \Lambda = ga \in GF(p)\}$	Poincaré group
$\mathscr{D} = \{(\Lambda, a) \mid \tilde{\Lambda} g \Lambda = ga \in GF(p)$	Dieudonne group
$\mathscr{P} = \mathscr{L} \otimes \mathscr{T}$	$\mathscr{D} = \mathscr{C} \otimes \mathscr{T}$
$\mathscr{L} = \mathscr{L}_0 \otimes \mathscr{I}$	$\mathscr{P} = \mathscr{P}_0 \otimes \mathscr{I}$
$\mathscr{C} = \mathscr{C}_0 \otimes \mathscr{I}$	$\mathscr{D} = \mathscr{D}_0 \otimes \mathscr{I}$

where

$$\mathscr{I} \equiv \{1, P, T, PT\}$$

4. *Reflections in the Light Cone*

Normally $\mathscr{L} = \mathscr{C}$ and $\mathscr{P} = \mathscr{D}$ but for a finite field $GF(p)$ Dieudonne has shown that this is not the case. The matrices Λ such that

$$\tilde{\Lambda} g \Lambda = -g$$

can be generated by the Lorentz matrices together with a particular class of non-Lorentz matrices, e.g.

$$\Lambda_0 = \begin{bmatrix} 0_q & 0_q & 0_q & -1_q \\ 0_q & -\alpha_q & \beta_q & 0_q \\ 0_q & \beta_q & \alpha_q & 0_q \\ 1_q & 0_q & 0_q & 0_q \end{bmatrix} \qquad \text{where } {\alpha_q}^2 + {\beta_q}^2 = -1_q$$

$$(\Lambda_0, 0): x^2 \to -x^2$$

Since the light cone is preserved Coish maintains that these elements should be retained and the physically relevant group the Dieudonne rather than the Poincaré group. Such transformations $(\Lambda_0, 0)$ are referred to as reflections in the light cone. We shall see that the existence of elements α_q, β_q s.t. ${\alpha_q}^2 + {\beta_q}^2 = -1_q$ is instrumental in the downfall of this model.

5. *The Unimodular Group*

We must now consider how the normal local isomorphisms to the unimodular groups become modified.

The unimodular group is normally over the complex field and so we must first ask what we mean by 'complex' in the Galois case.

Just as the complex field is an extension field of the reals we take the most natural analogue to be the extension field of the $GF(q)$ by some 'non-square' element. For a K-prime -1_q is 'non square' so let us select it and construct the extension. If we do this by giving the normal structure to $GF(q) \times GF(q)$ we obtain a field of q^2 elements and by the uniqueness theorem for Galois fields we must have constructed $GF(q^2)$.

Every element $z \in GF(q^2)$ can thus be written

$$z = x + iy \qquad \text{where} \qquad x, y \in GF(q) \qquad \text{and} \qquad i^2 = -1_q$$

To define unimodular we need complex conjugation. We note that $p = 4l - 1$ for a K-prime with l an integer and it follows that

$$i^p = -i$$

Thus $(x + iy)^p = (x - iy)$ which follows as a trivial application of Fermats theorem $x^q \equiv x (\text{mod } q)$. We are then led to define $\bar{z} = z^p$

$$z^{p+1} = \bar{z}z = (x^2 + y^2)$$

But is should be realised that $\bar{z}z$ may not be 'square'.

One constructs the matrix corresponding to the space-time event exactly as before, $a \to a_0\sigma_0 + \sigma_i a_i$, and Lorentz transformations are induced in precisely the normal manner

$$a \to \tilde{a} = b^+ab$$

There are, however, some interesting new features.

The operation of 'strong reversal' $x \to -x$ for all the event coordinates becomes realised in this way also by matrices of the type

$$a(-I) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix} \qquad \text{where } \bar{\zeta}\zeta \equiv \zeta^{p+1} = -1_q$$

Also it is clear that all the matrices of the form $w^{\alpha}a(=0, \ldots p)$ where $w^{p+1} = 1$ are mapped on to the same matrix. We shall have more to say about this 'gauge freedom' in a while.

Finally we note that for the Coish group we require $\|a\|^{p+1} = -1$, e.g.

$$\begin{pmatrix} 0 & \zeta \\ -1 & 0 \end{pmatrix}$$

Once again this curious feature is related to the existence of an element $\zeta = \alpha + \sqrt{(-1_q)}\beta$ for which $\zeta\bar{\zeta} = \alpha_q{}^2 + \beta_q{}^2 = -1_q$.

6. *Representations*

The most important point here is the nature of the space on which the group is realised. On the one hand the most natural space might be thought to be $GF(q^2)$ and we would have the modular representations of the group. From the quantum mechanic outlook, however, such representations are probably of little value. If the Hilbert space formulation of quantum mechanics is correct, which may not be the case, and a ground field containing the real numbers as a subfield is sought, the choice of small. It is a well-known result that there are only two other possibilities, the complex field and the quaternions. Naturally each has been considered as a field for the physical Hilbert space. The analyses of Stueckelberg suggest that the real case is effectively equivalent to the complex case due to superselection rules that operate. However, there is to our knowledge no physical reason against a quaternionic Hilbert space other than the fact that the complex space has so far seemed to be sufficient and one does not needlessly invite complication. We seek, then, unitary irreducible representations of the relativity groups in a complex Hilbert space. It should be remembered that the quantities of primary interest from the group-dynamical angle are the Clebsch–Gordon coefficients which will now be complex valued.

7. *The Complex Case*

Since we have, as in the usual case, semi-direct product structure we can use the theory of induced representations. Let us commence with a reminder of how the theory is constructed. Let us denote the semi-direct product of two groups A and B where B is the invariant abelian subgroup, which is for us the translation group, by $A \otimes B$. We consider the dual space $\hat{B}$ of characters of B

which are maps $\chi : B \to C$ having the property that

$$\chi(b_1 + b_2, p) = \chi(b_1, p)\chi(b_2, p) \qquad |\chi(b, p)| = 1$$

where p parameterises $\hat{B}$.

If $\alpha : b \to \alpha(b)$ is the action of $\alpha \in A$ on B which defines the semi-direct product there is a natural action of A on $\hat{B}$ defined as follows:

$$\chi(b, \alpha^{-1}p) = \chi(\alpha(b), p))$$

Making the orbit decomposition of $\hat{B}$ associated with this action we get a well-defined division of B into invariant subspaces.

Let $p^i = U_\alpha \alpha(p_i)\alpha \in A$ be the orbits where i labels the equivalence classes. The 'stability group' of a point on an orbit p^i is defined to be the subgroup

$$S_p{}^i = \{\alpha \,|\, \chi(b, p) = \chi(\alpha(b), p)\}$$

Given an I.U.R. of $B \otimes A$ in a Hilbert space H

$$U : (\alpha, b) \to U(\alpha, b)$$

we select a basis in which the subgroup $U(1, b) = U(b)$ is completely reduced.

$$U(b)\,|\,p, \gamma\rangle = \chi(b, p)\,|\,p, \gamma\rangle$$

and γ is a degeneracy parameter.

$$U(b)U(\alpha^{-1}, 0)\,|\,p\gamma\rangle = U(\alpha^{-1}, 0)U(\alpha(b))\,|\,p, \gamma\rangle$$
$$= \chi(b, \alpha^{-1}p)U(\alpha^{-1}, 0)\,|\,p, \gamma\rangle$$

Thus an I.R. of $B \otimes A$ restricted to B contains all characters $\chi(b, p)$ of one certain orbit p^i. Clearly,

$$U(p, 0)\,|\,p, \gamma\rangle = \sum_{\gamma'} |\,p, \gamma'\rangle D_{\gamma'\gamma}(p) \qquad p \in S_p{}^i$$

There is a (1-1) correspondence of $p \in p^i$ with the left cosets of $S_p{}^i$ in A. Selecting a representative c from each coset, where

$$c(p, p^i)p^i = p$$

gives

$$U(b)U(c(p, p^i), 0)\,|\,p^i, \gamma\rangle = \chi(b, p)U(c(p, p^i), 0)\,|\,p^i, \gamma\rangle$$

We can define the degeneracy parameter so that

$$U(c(p, p^i), 0)\,|\,p^i, \gamma\rangle = |\,p, \gamma\rangle$$

The element $p(\alpha, p) = c^{-1}(\alpha(p), p^i)\alpha c(p, p^i) \in S_p{}^i$ and so

$$U(\alpha, 0)\,|\,p, \gamma\rangle = U(c(\alpha(p), p^i), 0)U(p(\alpha, p))U(c^{-1}(p, p^i), 0)\,|\,p, \gamma\rangle$$
$$= \sum_{\gamma'} |\alpha(p), \gamma'\rangle D_{\gamma'\gamma}(p(\alpha, p))$$

Clearly $U(\alpha, 0)$ is irreducible if D is.

Summarising,

$$U(1, b)\,|\,p, \gamma\rangle = \chi(b, p)\,|\,p, \gamma\rangle \qquad p \in p^i$$
$$U(\alpha, 0)\,|\,p, \gamma\rangle = \sum_{\gamma'} |\,\alpha(p), \gamma'\rangle D_{\gamma'\gamma}(p(\alpha, p))$$

The problem has been reduced then to the representation theory of the various stability groups. This was so far as Ahmavaara could go in 1965 as the representations over the complex field were not explicitly available at that time (Ahmavaara, 1965). He could only guess at their general structure. For example he noted that the complex values of $c'(p)$, the covering group of the Coish group required to produce single-valued representations, would have the form

$$(\Lambda p, 0) \to \exp\,[2\pi i Qk/q + 1]D^{(s)}(\Lambda p, 0) \qquad k = 0, 1, 2, \ldots q$$

where the matrices $D^{(s)}(\Lambda(p, 0))$ give a univalent irreducible unitary representation of the little Coish group $c(p)$ for every value of s. This space has presented us in a very natural, intrinsic, way with a conserved quantum number which can take integral multiples of a single value. This quantum number, interpreted naturally as charge, is one of the most charming features of this model.

8. *The New Representations*

Since the representations that Ahmavaara lacked have since become available this seems a suitable time to consider the consequences. In fact only the representation of $SL(2, GF(p))$ are available (Tanaka, 1967; Silberger, 1969) but the following facts show that these are sufficient for our immediate purpose.

The matrices of $SU(2, GF(p^2))$ have the form

$$Y = \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} \qquad \alpha, \beta \in GF(p^2)$$

considered as a subgroup of $GL(2, GF(p^2))$, it is similar to the subgroup $SL(2, GF(p))$, e.g. the map $\eta : SU(2, GF(p^2)) \to SL(2, GF(p))$ defined by

$$\eta\left\{\begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}\right\} = X\begin{bmatrix} \alpha & \alpha \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}X^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where

$$X = \begin{bmatrix} 1 & \zeta \\ \sqrt{2} & -\sqrt{(2)}\bar{\zeta} \end{bmatrix}$$

and, as a result,

$$a = \tfrac{1}{2}(\alpha + \bar{\alpha} - \bar{\beta}\zeta - \bar{\zeta}\beta)$$

$$b = \frac{1}{2\sqrt{\nu}}(\alpha - \bar{\alpha} + \bar{\zeta}\beta - \bar{\beta}\zeta)$$

$$c = \frac{1}{2\sqrt{\nu}}(\alpha - \bar{\alpha} - \bar{\zeta}\beta + \bar{\zeta}\beta)$$

$$d = \tfrac{1}{2}(\alpha + \bar{\zeta}\beta + \bar{\alpha} + \bar{\zeta}\beta)$$

and ν is the non-square element which determines the extension of $GF(p)$ to $GF(p^2)$, $\zeta\bar{\zeta} = -1$ and η is an isomorphism. Consequently the two groups are isomorphic.

Similarly we find that the subgroup of matrices of the form

$$\begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} \qquad \alpha, \beta \in GF(\beta)$$

which we shall call $SU(1, 1; GF(p^2))$, is also isomorphic to $SL(2, GF(p))$, e.g. the map

$$\xi : SU(1, 1; GF(p^2)) \to SL(2, GF(p))$$

defined by

$$\xi\left\{\begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix}\right\} = W\begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix}W^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where

$$W = \begin{bmatrix} 1 & 1 \\ \sqrt{\nu} & -\sqrt{\nu} \end{bmatrix}$$

and, as a result,

$$a = \tfrac{1}{2}(\alpha + \bar{\alpha} + \beta + \bar{\beta})$$

$$b = \frac{1}{2\sqrt{\nu}}(\alpha - \bar{\alpha} + \bar{\beta} - \beta)$$

$$c = \frac{\sqrt{\nu}}{2}(\alpha + \beta - \bar{\alpha} - \bar{\beta})$$

$$d = \tfrac{1}{2}(\alpha - \beta + \bar{\alpha} - \bar{\beta})$$

and as before ξ is an isomorphism.

On examination of the representations of Tanaka (1967) and Silberger (1969) one finds, as mentioned previously, that there are no low-dimensional irreducible representations at all. This should be compared with the modular representations of $SU(2, GF(p^2))$ to be found in Baltamelti & Blasi (1968). It is surprising that this space, seemingly so odd, should fall only at a final fence and come close to being an acceptable model at all.

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